KINETIC STABILITY OF CENTRAL ORBITS.*

BY PROFESSOR W. WOOLSEY JOHNSON.

1. In the chapter on Central Orbits of Tait and Steel's Dynamics of a Particle, Fourth Edition, p. 125, occurs an investigation of the apsidal angle of a nearly circular orbit. When the attraction varies inversely as the nth power of the distance, the expression found becomes imaginary when n exceeds 3, and the remark is made that "the investigation furnishes a simple example of the determination of the conditions of *Kinetic Stability*, which we cannot discuss in this elementary treatise." It may not be without interest to show that an investigation of a no less elementary character will furnish a satisfactory discussion of this interesting subject, so far as it relates to central forces.

2. The usual polar equation of the central orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}, \tag{1}$$

in which P is the attraction acting on a unit of mass.

The first integral of this equation found in the usual way is

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2}{h^2}\int \frac{P\,du}{u^2}.$$

Since $\int \frac{P\,du}{u^2} = -\int P\,dr = -V + C$, V being the potential function, this equation may be written

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2}{h^2}(C - V) - u^2 = \psi(u), \tag{2}$$

the function $\psi(u)$ depending not only on the given law of force, but also upon the values given to the two constants h and C.

3. In discussing the function $\psi(u)$ we have only to consider positive values of u, (the reciprocal of r the distance from the centre of force,) and it is evident from equation (2) that the values of u for every point of an actual orbit must be such as to make $\psi(u)$ positive, except the maxima and minima values

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which make $\psi(u) = 0$. Thus, if $\psi(u)$ is negative for all positive values of u, there is no orbit having the given values of C and h; and if $\psi(u)$ is always positive there is no maximum or minimum distance, but an orbit extending to the centre of force in one direction and to infinity in the other.

In the general case, putting $\psi(u) = 0$, we have the apsidal values of u. Let the positive roots of this equation be u_1, u_2, etc., which at first suppose to be all distinct. If we draw circles about the centre of force with the reciprocals of these quantities as radii, the plane will be separated into spaces in which $\psi(u)$ is alternately positive and negative (for $\psi(u)$ cannot become infinite for a finite value of u). In each of the spaces where $\psi(u)$ is positive an orbit exists with the given values of C and h, having its apsides upon the circle or circles which bound the space.

4. Now suppose the values of C and h to be so related that $\psi(u) = 0$ has a pair of equal roots, say u_0. As they approach the special values which fulfil the condition, $\psi(u) = 0$ will have two roots near to u_0. If in the annular space between the corresponding circles $\psi(u)$ is positive, there exists an orbit which, as the annular space contracts, approaches the circular form, and finally becomes a circular orbit described like any other orbit with kinetic stability. The differential equation is satisfied by the constant value $u = u_0$, and for all neighboring values of u $\psi(u)$ is negative.

If, on the other hand, $\psi(u)$ is negative in the narrow annular space, there are no orbits approximating the circular form. $u = u_0$ still satisfies the differential equation, but $\psi(u)$ is positive for neighboring values of u on either side, and when $u = u_0$ the body may be regarded as being at an apse of either of the orbits which exist in the two spaces in which $\psi(u)$ is positive, which are now brought into juxtaposition. The circular orbit is only a singular solution, and not a case of the general integral of the differential equation, and is described with *Kinetic Instability*.

5. To fix the ideas as to the mode in which the condition of equal roots is approached, as well as to obtain convenient expressions for the condition and for the criterion of stability, let us assume u_0 as a possible value of u, say that of a point of projection, and then express C and h in terms of two other parameters. For this purpose take v_0, the initial velocity, and γ, the angle between the direction of the initial radius vector r_0 and that of projection.

Observe that C denotes the total energy of a unit mass in the orbit and that h is the double area described by the radius vector in a unit of time. Hence

$$C = V_0 + \tfrac{1}{2}v_0^2, \qquad h = r_0 v_0 \sin\gamma, \tag{3}$$

and $\psi(u)$ becomes

$$\psi(u) = \frac{2u_0^2}{v_0^2 \sin^2 \gamma}(V_0 - V) + \frac{u_0^2}{\sin^2 \gamma} - u^2,$$

which makes $\psi(u_0)$ positive. Now supposing C, and therefore v_0, to be fixed, h is restricted to be not greater than $r_0 v_0$, which is its maximum value, corresponding to $\gamma = 90°$ and making u_0 an apsidal value. The value of $\psi(u)$ when the point of projection is an apse is therefore

$$\psi(u) = \frac{2u_0^2}{v_0^2}(V_0 - V) + u_0^2 - u^2, \tag{4}$$

which makes $\psi(u_0) = 0$.

6. If now C (and v_0) has such a value that when $\gamma = 90°$ we reach the case of equal roots, we must have $\psi'(u_0) = 0$. Differentiating equation (4),

$$\psi'(u) = \frac{2u_0^2}{v_0^2}\left(-\frac{dV}{du}\right) - 2u;$$

or, since $-\frac{dV}{du} = \frac{P}{u^2}$,

$$\psi'(u) = \frac{2u_0^2 P}{v_0^2 u^2} - 2u, \tag{5}$$

and

$$\tfrac{1}{2}\psi'(u_0) = \frac{P_0}{v_0^2} - u_0.$$

For equal roots we must therefore have (as well as $\gamma = 90°$,)

$$v_0^2 = P_0 r_0; \tag{6}$$

in other words, the centrifugal force must equal the attraction at the initial point.

7. Passing now to the criterion of stability, observe that when the circular orbit is stable $\psi(u_0)$, of which the value is zero, is a maximum value of $\psi(u)$, and this requires $\psi''(u_0)$ to be negative.

Differentiating (5) and putting $P_0 r_0$ for v_0^2 by (6), we find

$$\psi''(u) = \frac{2u_0^2}{P_0}\left[\frac{1}{u^2}\frac{dP}{du} - \frac{2P}{u^3}\right] - 2,$$

whence

$$\tfrac{1}{2}\psi''(u_0) = \frac{u_0}{P_0}\left[\frac{dP}{du}\right]_0 - 3.$$

It follows that, while with any law of central attraction a circular orbit is possible with any radius r_0, it will be $\left\{\begin{matrix}\text{stable}\\\text{unstable}\end{matrix}\right\}$ according as $\left[\frac{u}{P}\frac{dP}{du}\right]_0$ is $\left\{\begin{matrix}\text{less}\\\text{greater}\end{matrix}\right\}$ than 3.*

8. The case in which $P = \mu u^3$ is peculiar, since the criterion is then identically equal 3. The special case occurs when $C = 0$, the orbit being an equiangular spiral unless $h^2 = \mu$, which makes $\gamma = 90^\circ$, when it becomes a circle, and the circle must be regarded as described with kinetic instability.

LAGRANGE'S PLACE IN THE THEORY OF SUBSTITUTIONS.†

BY DR. JAMES PIERPONT.

IN the present brief note I cannot vindicate Lagrange's right to the title of creator of the theory of substitutions; but I hope, by presenting a few examples of his methods, to show the importance of considering him from this point of view. Lagrange was led to the study of this theory by his attempts to solve equations of degree higher than the fourth. Speaking of the inherent difficulties which this thorny subject offered to the investigator, he remarks:‡

"The theory of equations is of all parts of analysis the one, we would think, which ought to have acquired the greatest degree of perfection, by reason both of its importance and of the rapidity of the progress that its first inventors made; but although the mathematicians of later days have not ceased to apply themselves, there remains much in order that their efforts may meet with the success that one could desire. In regard to the resolution of literal equations one has hardly advanced further than one was in Cardan's time, who was the first to publish the resolution of equations of the third and fourth degree. The first successes of the Italian analysts in this branch seem to have marked the limit of possible discoveries: at least it is certain that all attempts that have been made up to the present to push back the limits of this branch of algebra have hardly served for other purposes than

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* An equivalent criterion is otherwise derived in Thomson and Tait's Natural Philosophy, § 350.

† Read before the Yale Mathematical Club.

‡ Lagrange: Nouveaux Mémoires, Acad. Sciences Berlin, years 1770–71. Also, Œuvres, vol. III, pp. 205–421, Réflexions sur la résolution algébrique des équations.

to find new methods to solve the equations of third and fourth degree, none of which seem applicable to equations of higher degrees."

In his great paper published in the Mémoires of the Academy of Sciences at Berlin in the years 1770–71, under the title "Réflexions sur la résolution algébrique des equations," Lagrange proposed to examine the different methods which one had found up to then to solve algebraic equations, to reduce them to general principles, and to show *à priori* why these methods succeeded in case of the cubic and biquadratic, but failed for equations of higher degree. To do this Lagrange took up successively the various methods proposed by Cardan, Ferrari, Descartes, Tschirnhaus, Euler, and Bézout, and showed that the roots of the various resolvents upon whose solution the solution of the given equation depended were *rational* functions of the roots of the given equation. Here, then, was a great and fundamental step in advance. The problem of the solution of equations was shown to depend upon the properties of *rational* functions of the roots. To study the properties of these functions, Lagrange invented a "*calcul des combinaisons*," as he styled it, which was nothing else than the first rudiments of the theory of substitutions.

By means of this new *calcul* Lagrange was placed in a position to tell in advance the result and character of certain investigations, in much the same way that algebra serves for numerical problems. Lagrange himself characterizes his method in the following words:

"These, then, if I mistake not, are the true principles for the resolution of equations. The analysis is reduced, as is seen, to a species of calculus of combinations by means of which one finds *à priori* the results one should expect." Lagrange gives the new calculus a broad and solid basis. Among the various theorems he established for rational functions of the roots of a general equation of nth degree, one is of fundamental importance: a function V which takes on $n!$ values for the substitutions of the symmetric group is root of an irreducible equation of degree $n!$ whose coefficients are rationally known. The roots of this equation are rational functions of one another, and possess the remarkable property that every rational function of the roots of the given equation can be expressed rationally in one of them, and hence the roots themselves. Similar functions he defines as those having the same group. Two similar functions are rationally expressible by each other. A rational function of the roots which takes on ρ values for the symmetric group, is root of an equation of degree ρ, whose coefficients are rational in the coefficients of the given equation. Further, ρ is a divisor of $n!$

If ϕ and ψ take on respectively $m\rho$ and ρ values for the symmetric group, then ϕ is a root of an equation of mth degree whose coefficients are rational in ψ. The function ψ can be expressed rationally in ϕ.

Let us see how such theorems as these enable Lagrange to assign *à priori* the reason for the success of the various methods proposed to solve the cubic and biquadratic, and their failure when applied to equations of higher degrees. I have already remarked that Lagrange found that the resolving functions employed by his predecessors were rational. In final analysis, he found that they all belonged to the type

$$t = x_1 + \alpha x_2 + \alpha^2 x_3 + \ldots + \alpha^{n-1} x_n,$$

where $x_1, x_2, \ldots x_{n-1}$ are the roots of the given equation $f(x) = 0$ and α is an imaginary nth root of unity. Let us see to what equations these functions lead. Two cases present themselves according as n is prime or composite. Let first n be prime.

Then the function

$$\theta_1 = t_1^n = (x_1 + \alpha x_2 + \ldots + \alpha^{n-1} x_n)^n$$

remains unchanged for the cyclic substitutions

$$| z, z + a | \pmod n \quad a = 0, \quad 1 \ldots n - 1.$$

For the substitutions

$$| z, bz | \pmod n \quad b = 1, 2 \ldots n - 1,$$

θ_1 takes on $n - 1$ values

$$\theta_1, \theta_2 \ldots \theta_{n-1},$$

obtained by replacing α by respectively $\alpha^2, \alpha^3, \ldots$

Consider the equation

$$\theta = (\theta - \theta_1) \ldots (\theta - \theta_{n-1}) = 0.$$

Its coefficients are symmetrical functions of $\theta_1, \theta_2, \ldots$. Let ψ be such a function. It is root of an equation of degree $\rho = (n - 2)!$ If ψ can be found in any way, the coefficients of $\theta = 0$ being rational in ψ, are rationally known, and the solution of $f(x) = 0$ depends now upon an equation of degree $n - 1$.

When $n = 3$, $\rho = 1$; $n = 5$, $\rho = 6$.

Thus for the cubic we see that the coefficients of $\theta = 0$ are rational, and the solution depends therefore upon an equation of the second degree only. As soon, however, as the prime

$n > 3$, the coefficients depend upon an equation of degree higher than the given equation. For $n = 5$ it is already of sixth degree. In passing I note that Lagrange by the considerations of his new *calcul* made the solution of the quintic depend upon a sextic. The methods of Tschirnhaus, Euler, and Bézout lead to equations of twenty-fourth degree. Lagrange sought in vain to find a resolving function which should satisfy an equation of degree less than five.

It is not uninteresting also to remark, that whenever an equation of prime degree is algebraically soluble, Lagrange's method leads us directly to the solution. When $n = 5$, this was already noticed by Malfatti, a contemporary of Lagrange.

When n is a composite number, the foregoing considerations do not hold.

Let $n = p\nu$ where p is a prime factor.

Lagrange writes the roots in the array

$$\begin{array}{cccccc} x_1 x_2 & \cdot & \cdot & \cdot & \cdot & x_\nu \\ x_{\nu+1} & \cdot & \cdot & \cdot & \cdot & x_{2\nu} \\ x_{2\nu+1} & \cdot & \cdot & \cdot & \cdot & x_{3\nu} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{(p-1)\nu+1} & \cdot & \cdot & \cdot & \cdot & x_{p\nu} \end{array}$$

Let the equation whose roots form the ith row be $\phi_i = 0$. The coefficients of this equation are symmetric functions of the elements of the corresponding row.

If $X_i = x_{(i-1)\nu+1} + x_{(i-1)\nu+2} + \ldots x_{i\nu}$, $i = 1, 2 \ldots p$, the coefficients of $\phi_i = 0$ are rational in X_i, and when X_1, $X_2 \ldots$ are known, the coefficients of $\phi_1 = 0$, $\phi_2 = 0 \ldots$ are rationally known, and the solution of an equation of degree $p\nu$ is reduced to the solution of p equations of degree ν. If now ν be composite we may break ν into two factors, $\nu = p_1 \nu_1$, p_1 being prime, and proceed as before.

Let us now return to the determination of the quantities X_1, $X_2 \ldots$ which are roots of an equation of prime degree, namely,

$$X = (X - X_1)(X - X_2) \ldots (X - X_p) = 0,$$

whose coefficients are roots of a rational equation of degree

$$\rho = \frac{n!}{(\nu!)^p p!}.$$

The equation $X = 0$ being prime, may be solved by the foregoing method.

When

$$n = 4, \ \rho = 3; \quad n = 6, \ \rho = 10 \text{ or } 15.$$

Here again Lagrange's methods gave him a clear insight into the reason for the success and failure of his predecessors' methods, according as $n \lessgtr 4$ or > 4.

Particularly instructive and important is the application Lagrange made of his methods to the equations upon which the division of the circumference of the circle into n equal parts depends,* for it would have required a far less attentive reader than Lagrange's illustrious disciple, Abel, not to have perceived what slight modifications were necessary in order to apply Lagrange's methods to the corresponding equations in the theory of elliptic functions.

The equations in question have the form

$$x^{n-1} + x^{n-2} + \ldots + x + 1 = 0$$

where we suppose n prime. Lagrange proceeds as follows: Let $n - 1 = pq$, p prime. If r be an imaginary nth root of unity and α a primitive congruence root for n, we can arrange the pq roots thus:

$$\begin{array}{cccccc} r & r^{\alpha^p} & r^{\alpha^{2p}} & \ldots & r^{\alpha^{(q-1)p}} \\ r^{\alpha} & r^{\alpha^{p+1}} & r^{\alpha^{2p+1}} & \ldots & r^{\alpha^{(q-1)p+1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ r^{\alpha^{p-1}} & r^{\alpha^{2p-1}} & r^{\alpha^{3p-1}} & \ldots & r^{\alpha^{pq-1}} \end{array}$$

If $X_1, X_2, \ldots$ denote, as in the general case just treated, the sums of the elements of the various rows, Lagrange showed that the equation

$$X = (X - X_1)\ldots(X - X_p) = 0$$

is rational and can be algebraically solved, so that the solution of the original equation depends upon the solution of p equation of degree q, and so forth. To solve the equation $X = 0$, he employs as in the general case the resolving function

$$\theta_1 = t_1^p = (X_1 + \alpha X_2 + \ldots + \alpha^{p-1} X_p)^p.$$

But this quantity is here rationally known (α being supposed known), since, for any substitution which changes r into r^{α^s}, X_1 goes over into X_2, X_2 into X_3, etc.; thus θ_1 is unaltered. Developing and arranging according to powers of α, we have

$$\theta_1 = \xi_0 + \alpha\xi_1 + \ldots + \alpha^{p-1}\xi_{p-1},$$

* LAGRANGE: Traité de la résolution des équations numériques de tous les degrés. Paris, 1808. pp. 275–311.

where the ξ's are unchanged for r, r^α. But the ξ's being rational functions of r, r^α, r^{α^2} . . . , we have, e.g.,

$$\xi_0 = A + Br + Cr^\alpha + \ldots + Nr^{\alpha^{n-2}}.$$

As this is unchanged for r, r^α, we have by comparison

$$B = C = \ldots = N, \quad \text{or}$$

$$\xi_0 = A + B(r + r^\alpha + \ldots) = A - B;$$

that is, ξ_0 is known. As θ_1 is thus known, we get

$$X_1 + \alpha X_2 + \ldots + \alpha^{p-1} X_p = \sqrt[p]{\theta_1};$$

similarly $\quad X_1 + \beta X_2 + \ldots + \beta^{p-1} X_p = \sqrt[p]{\theta_2};$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$X_1 + \omega X_2 + \ldots + \omega^{p-1} X_p = \sqrt[p]{\theta_{p-1}};$$

also $\quad X_1 + X_2 + \ldots + X_p = -1,$

where $\alpha, \beta \ldots \omega$ are the $p-1$ imaginary pth roots of unity.

This system of linear equations gives us X_i; for example,

$$X_1 = \frac{-1 + \sqrt[p]{\theta_1} + \sqrt[p]{\theta_2} + \ldots + \sqrt[p]{\theta_{p-1}}}{p}.$$

The roots of the equation

$$x^{n-1} + x^{n-2} + \ldots + x + 1 = 0$$

are rational functions of one of them, x_0:

(1) $\qquad x_i = \theta_i(x_0), \qquad i = 1, 2 \ldots n-1.$

They enjoy further the property that

(2) $\qquad \theta_i \theta_\kappa x_0 = \theta_\kappa \theta x_0.$

But just these properties (1), (2) are enjoyed by the n^2 roots of the equation $F(x) = 0$ for dividing the argument of snz, namely,

$$x_{p,q} = \operatorname{sn}\left(\frac{z}{n} + \frac{4pK + 4qiK'}{n}\right),$$

when we consider the quantities k^2, snz, cnz, dnz, and

$$y_{p,q} = \mathrm{sn}\left(\frac{4pK + 4qiK'}{u}\right),$$

as known.

How closely Abel follows Lagrange in his solution of equations of the type $F(x) = 0$ is shown by the following sketch of his method :*

Let $f(x) = 0$ be any (irreducible) equation whose roots enjoy the properties (1), (2). We may represent them by the array

$$\begin{array}{lllll} x_0 & \theta x_0 & \dots & \theta^{n-1}x_0; \\ x_1 & \theta x_1 & \dots & \theta^{n-1}x_1; \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{m-1} & \theta x_{m-1} & \dots & \theta^{n-1}x_{m-1}. \end{array}$$

Consider the equation $\phi(x) = 0$, whose roots are the elements of the first row. If we suppose the coefficients of this equation to be known, it is soluble. In fact setting with Lagrange

$$\psi_0 = (x_0 + \alpha\theta x_0 + \dots + \alpha^{n-1}\theta^{n-1}x_0)^n,$$

ψ_0 remains unchanged for $| x_0, \theta x_0 |$, as in the case of the cyclotomic equations just considered.

But ψ_0 is rational, for if we denote by ψ_m what ψ_0 becomes after the substitution $| x_0, \theta^m x_0 |$, since $\psi_m = \psi_0$, we have $\psi_0 = \psi_1 = \dots \psi_{n-1}$, whence

$$\psi_0 = \frac{1}{n}(\psi_0 + \psi_1 + \dots + \psi_{n-1}),$$

a symmetric function of the roots of $\phi(x) = 0$.

Thus, precisely as before, we have a system of linear equations which gives, for example,

$$x_0 = -\frac{A}{n} + \frac{\sqrt[n]{\psi_0} + \sqrt[n]{\psi_1} + \dots + \sqrt[n]{\psi_{n-1}}}{n},$$

where A is the coefficient of x^{n-1} in $\phi(x) = 0$.

We return to the equation upon which the coefficients of $\phi(x) = 0$ depend.

* ABEL: Mémoire sur une classe particulière d'équations résolubles algébriquements. *Crelle*, vol. 4. Also, Œuvres, 2d ed., vol. 1, pp. 478–507.

Let X_1 be the oft-considered symmetric function of the elements of the first row,

$$X_1 = x_0 + \theta x_0 + \ldots + \theta^{n-1} x_0, \text{ etc.}$$

The equation

$$X = (X - X_1)(X - X_2) \ldots (X - X_m) = 0$$

is rational. In fact

$$X_1 = F(x_0) = F(\theta x_0) = \ldots = F(\theta^{n-1} x_0).$$

Similarly $X_2 = F(x_1) = F(\theta x_1) = \ldots = F(\theta^{n-1} x_1)$.

. . . . . .

Hence $$X_1^\kappa = \frac{1}{n}\{(F(x_0))^\kappa + \ldots + (F(\theta^{n-1} x_0))^\kappa\};$$

$$X_2^\kappa = \frac{1}{n}\{(Fx_1)^\kappa + \ldots + (F(\theta^{n-1} x_1))^\kappa\},$$

. . . . . .

and thus $X_1^\kappa + X_2^\kappa + \ldots + X_m^\kappa$ is rational.

Now the equation $X = 0$, having the same properties as the original equation $f(x) = 0$, this last is algebraically soluble, and we have the theorem that the equation upon which the division of the argument of the elliptic function $\mathrm{sn}(z)$ depends is (under the previous assumptions) algebraically soluble.

Leaving Abel now, I must pass on to a last and even more striking example of the wonderful powers of Lagrange's "*calcul*" to announce *à priori* the results which one should expect. Lagrange, as I have remarked, had vainly endeavored to find a rational resolving function for the quintic which would satisfy an equation of degree less than five, and so place one in a position to effect the solution of this celebrated equation. Ruffini, an Italian contemporary of Lagrange, and his ardent disciple, succeeded by Lagrange's own methods in proving that no such function existed: in fact he demonstrated quite generally that no rational function of n elements existed which took on three or four values for the symmetric group, n being > 4. But Ruffini was too convinced of the latent power of his great countryman's methods to stop here: he boldly undertook* by their means to prove that the alge-

* P. Ruffini: Reflessioni intorno alla soluzione delle equazioni algebraiche generali. Modena. 1813.

braical solution of the general equation of degree > 4 was impossible. Although not altogether successful in his attempt, I wish to show with what simple means he did prove that if the expression for a root can be given such a form that the radicals in it are rational functions of the roots of the given equation, then the algebraical solution is impossible when the degree of the equation surpasses four.

In fact such an expression for the root could always be arrived at as follows: Let A_1 be a rational quantity, and let n_1 be a prime; then $P_1^{n_1} = A_1$ defines a first irrationality. Let A_2 be any rational function of quantities originally rational and P_1; then $P_2^{n_2} = A_2$ defines a second irrationality. Continuing in this way, any root of the given equation has the form $x_1 = A$, where A is a rational function of $P_1, P_2 \ldots$

Let now s be the cyclic substitution $s = (1\,2\,3\,4\,5)$, and let $P_s, P_{s^2}, P_{s^3}, P_{s^4}$ be the values of P_1 for s, s^2, s^3, and s^4, respectively. Then $P_s = \beta P_1$ where $\beta^{n_1} = 1$. Operating with $s, s^2 \ldots$ this gives

$$P_{s^2} = \beta^2 P_1, \quad P_{s^3} = \beta^3 P_1, \quad P_{s^4} = \beta^4 P_1; \quad \therefore \ \beta^5 = 1.$$

Similarly let P_σ be the value of P_1 for $\sigma = (1\,2\,3)$; then $P_\sigma = \gamma P_1$, and thus $\gamma^3 = 1$. But $P_{s\sigma} = \beta\gamma P_s$; hence, since $(s\sigma)^5 = 1$, $\beta^5\gamma^5 = 1$; $\therefore \gamma = 1$. Similarly, if $\rho = (3\,4\,5)$, then $P_\rho = P_{\rho^2} = P_1$. Now $\rho\sigma = s$; hence $P_{\rho\sigma} = P_s = P_1$. But $P_s = \beta P_1$; thus $\beta = 1$. Hence P_1 remains unaltered for s, thus also P_2, etc.; hence finally A. Thus the right hand of $x_1 = A$ is unchanged for s while the left-hand side is changed. Thus it is impossible to solve algebraically the general equation whose degree surpasses four.

The limited time at my disposal has not permitted me to discuss Lagrange's claims in detail; but the few examples I have chosen from Lagrange himself and from his immediate disciples will show, I think, how incomparably superior his methods were to those of his predecessors, Hudde, Saunderson, Le Sœur, and Waring; and it would be no difficult or ungrateful task to show how easily the ideas of Galois spring from the same source that inspired Ruffini, Cauchy, and Abel

NEW HAVEN, CONN.

GAUSS'S THIRD PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA.

BY PROFESSOR MAXIME BÔCHER.

IT is hoped that the following note may be of interest to some readers of the BULLETIN as indicating the connection between Gauss's third proof that every algebraic equation has a root (*Ges. Werke*, vol. III, p. 59 and p. 107), and those branches of mathematics which have since been developed under the names of the Theory of Functions and the Theory of the Potential. The considerations which follow have doubtless suggested themselves to other readers of Gauss's proof, and it even seems extremely probable that Gauss was led to the discovery of his proof by some method not very different from that here indicated.

Each of the three paragraphs of the present note which are marked with a roman numeral contains a complete proof of the theorem that every algebraic equation has a root. These proofs are arranged in the order of increasing complexity of detail, but of decreasing number of theorems assumed known. The last is essentially Gauss's proof.

Let $f(z) = 0$ be the equation (of the nth degree) for which we wish to prove the existence of a root, and suppose that in the polynomial $f(z)$ the coefficient of z^n is 1. The idea which underlies the proof we shall give is, that if we can prove that $\phi(z)/f(z)$, where $\phi(z)$ is a polynomial, does not remain finite for all values of z, $f(z) = 0$ must have a root. In what follows I let $\phi(z) = zf'(z) = zdf(z)/dz$. We will write $F(z) = \dfrac{zf'(z)}{f(z)} = u(x, y) + iv(x, y)$, where $z = x + yi$. Before taking up the various forms of proof we will note two points: 1st, $u(o, o) = 0$; 2d, if we describe a circle of radius a about the origin, $u(x, y)$ can be made positive at all points on the circumference of this circle by taking a sufficiently large, since $F(\infty) = n$.

Proof I.—$F(z)$ is obviously a monogenic function so that its real part $u(x, y)$ satisfies Laplace's equation. Moreover, if $f(z) = 0$ had no root, u would be finite, continuous, and single valued together with its first derivatives throughout the entire z-plane. We could therefore apply to it the proposition (familiar from the theory of the potential) that the average value of u upon the circumference of any circle is equal to the value of u at the centre. The value of u at the origin, however is zero while the average value of u upon the

circumference of a sufficiently large circle with centre at the origin is positive. We are thus led to a contradiction if we assume that $f(z) = 0$ has no root.

This proof, while of extreme simplicity, presupposes a certain knowledge of the theory of Laplace's equation. The proof ordinarily given of the average value theorem we have just used is based upon Green's theorem. Only that special case of Green's theorem is necessary, however, which is known as Gauss's theorem, viz.:

$$\int \frac{\partial u}{\partial n} ds = 0,$$

the integral being taken around the boundary of a region in which u satisfies Laplace's equation, and is, together with its first derivatives, finite, continuous, and single valued, and in which n denotes the (say external) normal. To deduce our average value theorem for the circle of radius a from this we have merely to take as our path of integration a circle of radius $r \leqq a$ concentric with the given circle of radius a. Using polar coordinates, Gauss's theorem may be written

$$\int_0^{2\pi} \frac{\partial u}{\partial r} d\phi = 0,$$

from which follows

$$\int_0^a \int_0^{2\pi} \frac{\partial u}{\partial r} d\phi dr = 0.$$

If we now reverse the order of integration and indicate the value of u at the centre by u_0, and the values of u on the circle of radius a by u_a, we get

$$\int_0^{2\pi} [u_a - u_0] d\phi = 0,$$

from which the desired mean value theorem follows at once.

It remains then to prove Gauss's theorem. Instead of deducing this theorem from Laplace's equation we may deduce it more simply for our purposes by the use of conjugate functions. The whole proof of the fundamental theorem of algebra may then be stated as follows:

Proof II.—If $f(z) = 0$ had no root, both u and v would be everywhere finite, continuous, and single valued, together with

their first derivatives. Moreover, being conjugate functions they satisfy the relation

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial s},$$

n and s representing any two directions such that the angle from n to s is 90°.* We have then, the integrals being taken around a circle of radius r and with centre at the origin,

$$\int_0^{2\pi} \frac{\partial u}{\partial r} d\phi = \frac{1}{r} \int_0^{2\pi} \frac{\partial v}{\partial \phi} \cdot d\phi = 0,$$

from which we deduce

$$\int_0^{r'} \int_0^{2\pi} \frac{\partial u}{\partial r} d\phi dr = 0.$$

If we here reverse the order of integration and remember that u vanishes at the origin, we get

$$\int_0^{2\pi} u d\phi = 0,$$

the integral being taken around a circle with centre at the origin and with any radius a. But this is impossible since by taking a sufficiently large we can make u everywhere positive.

Finally, if we do not wish to make use of even the fundamental formulæ for monogenic functions, we may proceed as follows:

Proof III.—Let $f(z) = z^n + (a_1 + b_1 i) z^{n-1} + \ldots + (a_{n-1} + b_{n-1} i) z + a_n + b_n i = \sigma + \tau i$, and let $z f'(z) = \sigma' + \tau' i$. Then, letting $z = r(\cos \phi + i \sin \phi)$, we have:

$$\begin{aligned} \sigma &= r^n \cos n\phi + a_1 r^{n-1} \cos (n-1)\phi + \ldots \\ &\quad - b_1 r^{n-1} \sin (n-1)\phi - \ldots, \\ \tau &= r^n \sin n\phi + a_1 r^{n-1} \sin (n-1)\phi + \ldots \\ &\quad + b_1 r^{n-1} \cos (n-1)\phi + \ldots, \\ \sigma' &= n r^n \cos n\phi + (n-1) a_1 r^{n-1} \cos (n-1)\phi + \ldots \\ &\quad - (n-1) b_1 r^{n-1} \sin (n-1)\phi - \ldots, \end{aligned}$$

* The familiar formulæ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ merely express this fact for two special pairs of directions, and conversely the more general statement of the text can readily be deduced from these two formulæ.

$$\tau' = nr^n \sin n\phi + (n-1)a_1 r^{n-1} \sin (n-1)\phi + \ldots$$
$$+ (n-1)b_1 r^{n-1} \cos (n-1)\phi + \ldots,$$

$$F(z) = \frac{\sigma' + \tau' i}{\sigma + \tau i} = \frac{\sigma\sigma' + \tau\tau'}{\sigma^2 + \tau^2} + \frac{\sigma\tau' - \tau\sigma'}{\sigma^2 + \tau^2} i = u + vi.$$

We wish now to find the derivatives of u and v with regard to r and ϕ. For this purpose we note the following relations:

$$\frac{\partial\sigma}{\partial r} = \frac{\sigma'}{r}, \quad \frac{\partial\sigma}{\partial\phi} = -\tau',$$

$$\frac{\partial\tau}{\partial r} = \frac{\tau'}{r}, \quad \frac{\partial\tau}{\partial\phi} = \sigma'.$$

We also have formulæ of precisely the same sort for expressing the derivatives of σ' and τ' with regard to r and ϕ in terms of σ'' and τ'' where

$$\sigma'' = n^2 r^n \cos n\phi + (n-1)^2 a_1 r^{n-1} \cos (n-1)\phi + \ldots$$
$$- (n-1)^2 b_1 r^{n-1} \sin (n-1)\phi - \ldots,$$

$$\tau'' = n^2 r^n \sin n\phi + (n-1)^2 a_1 r^{n-1} \sin (n-1)\phi + \ldots$$
$$+ (n-1)^2 b_1 r^{n-1} \cos (n-1)\phi + \ldots$$

We get then by direct differentiation

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\phi}$$
$$= \frac{(\sigma^2+\tau^2)(\sigma\sigma''+\tau\tau'') + (\sigma\tau'-\tau\sigma')^2 - (\sigma\sigma'+\tau\tau')^2}{r(\sigma^2+\tau^2)^2} = T.$$

Now form the double integral

$$\Omega = \int_0^a \int_0^{2\pi} T d\phi dr.$$

If here we integrate first with regard to ϕ and then with regard to r, we obviously get $\Omega = 0$. If, however, we integrate first with regard to r and then with regard to ϕ, we get, remembering that u vanishes at the origin,

$$\Omega = \int_0^{2\pi} u d\phi,$$

the integral being taken around the circumference of a circle with radius a and centre at the origin, so that Ω will be posi-

tive if a is sufficiently large. The fact that we get different values for Ω according to the order of integration shows that T cannot be everywhere finite, continuous, and single valued, and this can be explained only by the vanishing of $\sigma^2 + \tau^2$ (since r, which also occurs in the denominator of T is a factor of each term of the numerator). A point where $\sigma^2 + \tau^2$ vanishes is a root of $f(z) = 0$.

In the proofs above given I have started with Gauss from the function $\frac{zf'(z)}{f(z)}$. There are, however, other functions which might have been used in almost exactly the same way, as for instance $\frac{z^n}{f(z)}$ and $\frac{1}{f(z)}$. In fact Gauss's proof would be somewhat simplified by the use of this last function.

HARVARD UNIVERSITY.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York, Saturday afternoon, April 27, at three o'clock. There were fourteen members present. In the absence of the president and vice-president, Professor Mansfield Merriman occupied the chair. On the recommendation of the council the following persons, nominated at the preceding meeting, were elected to membership: Professor Sara Antoinette Acer, Wells College, Aurora, N. Y.; Dr. Harris Hancock, University of Chicago, Chicago; Professor Munroe Benjamin Snyder, Central High School, Philadelphia. One nomination for membership was received. The following papers were presented:

(1) "On the derivation of the equations of rotation of bodies of variable form," by Professor R. S. WOODWARD.

(2) "A theory of mathematical methods," by Dr. E. M. BLAKE.

(3) "Kinetic stability of central orbits," by Professor W. WOOLSEY JOHNSON.

Professor Johnson's paper appears in the present number of the BULLETIN on page 193.

B. G. TEUBNER, of Leipsic, announces as in press the third volume of Dr. ERNST SCHRÖDER'S *Algebra der Logik;* it is devoted to the algebra and logic of relatives. The same publisher has in preparation an edition, in two 8vo volumes, of JULIUS PLÜCKER'S collected mathematical and physical papers. This publication is due to the initiative of the Göttingen Academy of Sciences. The first volume, which will be

adorned with a portrait of the author, will contain all the mathematical papers. Those of Plücker's works that have appeared in book form and are still obtainable (viz., Analytisch-geometrische Entwicklungen, System der analytischen Geometrie, Theorie der algebraischen Kurven, Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement) are excluded; but everything else has been carefully collected. Professor A. Schoenflies is the editor of this first volume, while the second, containing the physical papers, is under the care of Dr. F. Pockels.

In the February number of the *Intermédiaire des mathématiciens* (vol. 2, p. 110) Professor E. Lampe reports that Dr. Franz Meyer, in collaboration with other mathematicians, has undertaken the publication of a mathematical dictionary, or encyclopædia, in about six volumes. The project was discussed last year at the Vienna meeting of the German mathematical association, and will have the financial support of the academies of Vienna, Berlin, Munich, Göttingen, and Leipsic. B. G. Teubner will be the publisher.

The Department of Mathematics at Columbia College will give during the next academic year the following graduate courses, each occupying three hours a week throughout the entire year: (1) General theory of functions, by Professor T. S. Fiske; (2) Theory of substitutions, by Professor F. N. Cole; (3) Analytical theory of curves of double curvature and surfaces, by Mr. J. Maclay; (4) Ordinary and partial differential equations, by Mr. J. E. Hill; (5) Advanced differential and integral calculus, by Dr. J. B. Chittenden. The undergraduate work of the department will include courses in Elementary solid geometry, Algebra, Trigonometry, Analytical geometry, Projective geometry, Differential and integral calculus, Theory of equations, and Analytical geometry of three dimensions. These courses will be given by Dean Van Amringe, Professor Fiske, Mr. Maclay, Dr. Chittenden, and Mr. Hill. The courses in mathematics at Barnard College will be given by Professor Cole, Dr. Chittenden, and Mr. Hill.

Among those whom the mathematical world has recently lost through death are T. P. Kirkman, F.R.S., who died in London about the middle of February; Professor J. Worpitsky, who died in Berlin, March 8; Dr. E. D. F. Meissel, who died at Kiel, March 11; and Professor Schläfli, who died at Bern, March 20.

The following are the mathematical courses for the current summer semester at Berlin:—Professor Fuchs: Applications of

the elliptic functions; On the representation of functions which are defined by differential equations;—Professor Frobenius: Theory of determinants; Theory of algebraic equations (second part);—Professor Schwarz: Theory of the complex quantity; On curved surfaces and curves of double curvature; Introduction to the theory of analytical functions;—Professor Knoblauch: Integral calculus; Theory of partial differential equations of the first and second order; General theory of line systems;—Professor Hensel: Synthetic geometry; Differential calculus;—Professor Hettner: Dynamics; Mathematical exercises for new students;—Professor Lehmann-Filhés: Method of least squares;—Dr. Schlesinger: Plane and solid analytical geometry; On Gauss's investigations in the theory of elliptic functions;—Dr. Kötter: Analytical treatment of plane algebraic curves;—Dr. Hoppe: Integral calculus; Analytical mechanics;—Dr. Glan: Quaternions.

THE following are the mathematical courses for the current summer semester at Leipzig:—Professor Scheibner: Introduction to the infinite analysis;—Professor C. Neumann: Theory of potential and application to heat and electricity;—Professor Bruns: Mathematical geography, general astronomy and astrophysics; The solution of equations and theory of errors according to the method of least squares;—Professor Lie: Introduction to the theory of differential equations; Applications of the differential and integral calculus to geometry; Seminarium on group theory, differential invariants, and contact transformations;—Professor Mayer: Partial differential equations of the first order and Pfaff's problem; Introduction to algebra and the theory of determinants; Seminarium on the calculus of variations;—Professor Engel: Higher algebra;—Dr. Scheffers: Introduction to analytical geometry; Introduction to the theory of linear differential equations.

AMONG recent changes at Yale University we note that Dr. Irving Fisher, assistant professor of mathematics, has been, at his own desire, transferred to the department of political science. Dr. Fisher has done considerable work in the application of mathematics to economics. One of his papers on this subject was reviewed in the BULLETIN for June, 1893, p. 204. Dr. James Pierpont, a graduate of Yale and of the University of Vienna, has been appointed to an instructorship in the department of mathematics. His time will be divided equally between graduate and undergraduate courses.

DR. ERNST RITTER, privat-docent in mathematics at the University of Göttingen, has been appointed an assistant professor of mathematics at Cornell University. Dr. Ritter

took his doctor's degree at Göttingen in 1892. He has written some important papers on automorphic functions, which have been published in the *Mathematische Annalen*.

HARVARD UNIVERSITY. Besides regular undergraduate courses, the class-room work in which will amount to eighteen hours a week throughout the year, the following more advanced mathematical courses are offered for the year 1895–96:—By Professor J. M. Peirce: Quaternions (first course); Curves and surfaces; Linear associative algebra.†—By Professor Byerly: Modern geometry; Analytical mechanics.—By Professors Byerly and B. O. Peirce: Fourier's series, spherical harmonics, and the potential function.—By Professor B. O. Peirce: Hydromechanics.—By Professor Osgood: Differential and integral calculus (second course); Higher algebra; Theory of functions (second course).†—By Professor Bôcher: Theory of equations and invariants†; Infinite series, products, and continued fractions†; Theory of functions (first course); Picard's Traité d'Analyse, vol. I†; Functions defined by linear differential equations.†

The above courses will each consist of three lectures a week throughout the year except those marked †, which consist of one half that number of lectures. The following three courses, in which the work will consist of reading and investigation to be carried on by the students under the direction of the instructor, are also offered:—By Professor J. M. Peirce: Algebra of logic. By Professor Osgood: Klein's Ikosaeder. By Professor Bôcher: Lamé's products.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BESANT (W. H.). Conic sections treated geometrically. 9th edition, revised and enlarged. London, Bell, 1895. 8vo. 286 pp. 4s. 6d.

CHRISTENSEN (S. A.). Matematikens Udvikling i Danmark og Norge i det 18. Aarhundrede. Odense, 1895. 8vo. 270 pp. Mk. 5.40

DEL RE (A.). Lezioni di geometria proiettiva ed analitica. Vol. I., fasc. 1. Modena, Vincenzi, 1894. 8vo. 160 pp.

DUNAN (C.). Théorie psychologique de l'espace. Paris, 1895. 8vo. Fr 2.50

EBERHARD (V.). Die Grundgebilde der ebenen Geometrie. Vol. I. Leipzig, Teubner, 1895. 8vo. 48 and 302 pp. 5 plates. Mk. 14.00

EUCLIDIS Opera omnia. Ediderunt I. L. Heiberg et H. Menge. Vol. VII: Euclidis Optica, Opticorum recensio Theonis, Catoptrica, cum scholiis antiquis. Edidit I. L. Heiberg. Lipsiae, Teubner, 1895. sm. 8vo. 55 and 362 pp. Mk. 5.00

GRAY (A.) and MATHEWS (G. B.). A treatise on Bessel functions and their applications to physics. London and New York, Macmillan, 1895. 8vo. 291 pp. Cloth. $4.50

GUNDELFINGER (S.) Vorlesungen aus der analytischen Geometrie der Kegelschnitte, herausgegeben von F. Dingeldey. Mit einem Anhange, enthaltend Aufgaben und weitere Ausführungen. Leipzig, Teubner, 1895. 8vo. 8 and 434 pp. Mk. 12.00

HONTHEIM (J.). Der logische Algorithmus in seinem Wesen, in seiner Anwendung und in seiner philosophischen Bedeutung. Berlin, 1895. 8vo. 4 and 54 pp. Mk. 2.00

LEFFLER (A. C.). Sonja Kovalevsky, was ich mit ihr erlebte and was sie mir über sich selbst mitgetheilt hat. Aus dem Schwedischen übersetzt von H. v. Lenk. Leipzig, 1895. 12mo. 159 pp. Mk. 0.80

LACOUR (E.). Sur des fonctions d'un point analytique à multiplicateurs exponentiels ou à périodes rationnelles. [Thèse.] Paris, Gauthier-Villars, 1895. 4to. 53 pp.

LE ROUX (J.). Sur des integrales des équations linéaires aux dérivées partielles du second ordre à deux variables independantes. [Thèse.] Paris, Gauthier Villars, 1895. 4to. 95 pp.

NIEWENGLOWSKI (B.). Cours de géométrie analytique, à l'usage des élèves de la classe de mathématiques spéciales et des candidats aux écoles du gouvernement. Tome 2: Construction des courbes planes; Compléments relatifs aux coniques. Paris, Gauthier-Villars, 1895. 8vo. 296 pp. Fr. 8.00

PETERSEN (J.). Forelaesninger over Funktionsteori. Hefte I. Kjöbenhavn, 1895. 8vo. 112 pp. Mk. 3.90

PRADA (M. V.). Ecuaciones. La solución algebraica general y la numerica por fórmulas, demonstrada en el tercer grado por uno de sus siete modos y hasta en el grado n. Madrid, 1894. 8vo. 20 pp. Fr. 3.75

RÉPERTOIRE bibliographique des sciences mathématiques. 1re série. (Fiches 1 à 100.) Paris, Gauthier-Villars, 1894. Fr. 2.00

SCHENKEL (H.). Kritisch-historische Untersuchung über die Theorie der Gammafunction und der Euler'schen Integrale. [Diss.] Bern, Koerber, 1895. 8vo. 67 pp. 2 plates. Mk. 1.50

II. ELEMENTARY MATHEMATICS.

ANGEL (H.). Practical plane and solid geometry. Key to examinations of science and art department. (Subject I.) London, Chapman, 1895. 8vo. 272 pp. 3s. 6d.

BERTRAND (G.). Trattato di algebra elementare. Prima traduzione italiana con note ed aggiunte di E. Betti. 23a ristampa. Nuova edizione con modificazioni e aggiunte per cura di A. Socci. Firenze, 1895. Fr. 3.00

BÖTTGER (A.). Die ebene Geometrie. Für den Unterricht an der Realschule bearbeitet. Leipzig, Dürr, 1895. 8vo. 137 pp. Mk. 1.80

BRENNER (A.). 300 algebraische Aufgaben zur Lösung mittels einfacher Schlüsse, zunächst für Lehrerbildungs-Anstalten bearbeitet. 5te Auflage. Freising, Datterer, 1895. 8vo. 42 pp. Mk. 0.50

HOLZMÜLLER (G.). Methodisches Lehrbuch der Elementar-Mathematik. 3ter (Schluss-) Teil: Lehr- und Uebungsstoff zur freien Auswahl für die Prima realistischer Vollanstalten und höherer Fachschulen, nebst Vorbereitungen auf die Hochschul-Mathematik. Leipzig, Teubner, 1895. 8vo. 8 and 224 pp. Illustrated. Cloth. Mk. 2.80

HRABÁK (J.). Practische Hilfstabellen für logarithmische und andere Zahlenrechnungen. 3te Ausgabe. Leipzig, Teubner, 1895. 8vo. 5 and 253 pp. Mk. 3.00

PETERSEN (C. F.). Logarithme-Tabeller med fem Decimaler til praktisk Brug (uden Interpolation). 5. oplag. Christiania, 1894. 8vo. 109 pp. Boards. Mk. 1.20

REIDT (F.). Einleitung in die Trigonometrie und Stereometrie für die Untersekunda höherer Lehranstalten. 2te Auflage. Berlin, Grote, 1895. 8vo. 32 pp. Mk. 0.30

ROSSITER (W.). Algebra. (Local Examination Series.) New edition. London, Allman, 1895. 8vo. 226 pp. 1s. 6d.

VACQUANT (C.). Éléments de géométrie. 6e édition, revue, corrigée et conforme aux programmes du 15 juin 1891. Paris, Masson, 1894. 16mo. 591 pp.

III. APPLIED MATHEMATICS.

BERNOULLI (R.). Ueber den Einfluss des Lösungs-Mittels auf die elektromotorische Kraft. Leipzig, 1894. 8vo. 23 pp. Mk. 1.20

BRIGGS (W.) and BRYAN (G. H.). An elementary text-book of hydrostatics. (University Tutorial Series.) London, Clive, 1895. 12mo. 206 pp. 2s.

FORTSCHRITTE DER PHYSIK im Jahre 1889. Dargestellt von der Physikalischen Gesellschaft zu Berlin. Jahrgang 45. Abtheilung I: Physik der Materie, redigirt von R. Börnstein. Braunschweig, Vieweg, 1895. 8vo. 68 and 580 pp. Mk. 22.50

GLAZEBROOK (R. T.). Mechanics: an elementary text-book, theoretical and practical, for colleges and schools. Statics. (Cambridge Natural Science Manuals.) London and New York, Macmillan, 1895. 8vo. 180 pp. $1.00

GORE (J. H.). Geodesy. London, 1895. 8vo. Cloth. 5s.

GRAETZ (L.). Compendium der Physik. Für Studirende. 2te Auflage. Wien, Deuticke, 1895. 8vo. 2 and 454 pp. Mk. 7.00

HARKNESS (W.). On the magnitude of the solar system. Salem (Mass.), 1894. 8vo. 23 pp. $0.25

HERTZ (H.). Gesammelte Werke. Band I: Abhandlungen von 1879–84; Rede über Licht und Elektricität; Abhandlungen und Aufsätze neuesten Datums. Herausgegeben von P. Lenard. Leipzig, Barth, 1895. 8vo. 380 pp. 1 plate. Portrait. Illustrated. Mk. 12.00

——— Ueber die Beziehungen zwischen Licht und Elektricität. Vortrag. 9te Auflage. Bonn, Strauss, 1895. 8vo. 27 pp. Mk. 1.00

HOPPE (O.). Elementares Lehrbuch der technischen Mechanik. Abtheilung II: Mechanik der tropfbaren und gasförmigen Flüssigkeiten. Leipzig, Felix, 1895. 8vo. 11 and 135 pp. Illustrated: Mk. 4.50

IACOANGELI (O.). Triangolazioni topografiche e triangolazioni catastali. Modo di fondarle sulla rete geodetica, di rilevarle e calcolarle. Milano, 1895. 12mo. 215 pp. Con 4 quadri degli elementi geodetici, 32 modelli esemplificati pei calcoli trigonometrici e 32 incisioni. Fr. 8.00

KECK (W.). Vorträge über graphische Statik mit Anwendung auf die Festigkeits-Berechnung der Bauwerke, als Anhang zu des Verfassers "Vorträgen über Elasticitätslehre." Hannover, Helwing, 1895. 8vo. 7 and 99 pp. 4 plates. Mk. 3.50

LANGE (W.). Sammlung von mathematischen Formeln und Aufgaben zur Anwendung im Gebiete der Bautechnik. 2te Auflage. Heft 1 und 2. Bremen, 1895. 8vo. pp. 1 to 139. 2 plates. Mk. 2.90

LUEGER (O.). Lexikon der gesammten Technik und ihrer Hilfswissenschaften. Im Verein mit Fachgenossen herausgegeben. Vol. I. Stuttgart, Deutsche Verlags-Anstalt, 1895. 8vo. 6 and 800 pp. Half mor. Mk. 30.00

MEISSNER (G.). Die Hydraulik und die hydraulischen Motoren. 2te Auflage, von H. Hederich und Nowak. (In 25 Lieferungen.) Lieferung 1. Jena, Costenoble, 1895. 8vo. pp. 1 to 80. 5 plates. Mk. 3.00

RESAL (H.). Traité de mécanique générale comprenant les leçons professées à l'école polytechnique et à l'école des mines. 2me édition, entièrement refondue (en 7 volumes). Vol. I: Cinématique; théorèmes généraux de la mécanique; de l'équilibre et du mouvement des corps solides. Paris, Gauthier-Villars, 1895. 8vo. 20 and 303 pp. Fr. 6.50

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TOLOMEI (G.) e VESSICHELLI (G.). Elettricità e magnetismo, trattato teorico-pratico. Firenze, 1895. 8vo. 464 pp. Fr. 6.00

WINIAWSKI (L.). La méthode mathématique dans la sociologie et dans l'économie politique. Genève, 1895. 8vo. 15 pp. Fr. 1.00